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Indefinite Integrals

$$\int dx = x$$

$$\int af(x)dx = a\int f(x)dx$$

$$\int f(ax+b)dx = \frac{1}{a}\int f(u)du$$

$$\int f(x) \pm g(x)dx = \int f(x)dx + \int g(x)dx$$

$$\int \frac{f'(x)}{f(x)}dx = \log_e |f(x)|$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)}$$

$$\int e^x\left(f(x) + f'(x)\right)dx = e^x f(x)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \frac{1}{x}dx = \log_e |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\log_e a}$$

$$\int \log_e x dx = x\log_e x - x$$

Trigonometric integrals

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \csc^2 x dx = -\cot x$$

$$\int \sec x \tan x dx = \sec x$$

$$\int \csc x \cot x dx = -\csc x$$

$$\int \tan x dx = \log_e \sec x = -\log_e \cos x$$

$$\int \cot x dx = -\log_e \csc x = \log_e \sin x$$

$$\int \sec x dx = \log_e |\sec x + \tan x| = \log_e |\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)|$$

$$\int \csc x dx = -\log_e |\csc x + \cot x| = \log_e |\csc x - \cot x| = \log_e |\tan \frac{x}{2}|$$

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$$\int \sin^2 x dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2} \right)$$

$$\int \cos^2 x dx = \frac{1}{2}\left(x + \frac{\sin 2x}{2} \right)$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int \cot^2 x dx = -\cot x - x$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a} \right)$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log_e \left(\frac{x-a}{x+a} \right)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e \left(\frac{a+x}{a-x} \right)$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a} \right)$$

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \log_e | x + \sqrt{a^2 + x^2} |$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \log_e | x + \sqrt{x^2 - a^2} |$$

$$\int \sqrt{a^2 - x^2} dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a} \right)$$

$$\int \sqrt{a^2 + x^2} dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2} \log_e | x + \sqrt{a^2 + x^2} |$$

$$\int \sqrt{x^2 - a^2} dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e | x + \sqrt{x^2 - a^2} |$$

$$\int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x$$

$$\int \frac{dx}{1 + x^2} = \tan^{-1} x$$

$$\int \frac{dx}{\ } \quad \sec^{-1} x$$

$$\int \frac{dx}{x\sqrt{x^2+a^2}} = \frac{1}{a}\log_e \left| \frac{x}{a+\sqrt{x^2+a^2}} \right|$$

$$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\log_e \left| \frac{x}{a+\sqrt{a^2-x^2}} \right|$$

$$\int \frac{dx}{x(x^n+1)} = \log_e \left| \frac{x^n}{x^n+1} \right|$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$$

Hyperbolic integrals

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \tanh x dx = \log_e \cosh x$$

$$\int \coth x dx = \log_e \sinh x$$

$$\int \sec hx dx = 2\tan^{-1} e^x$$

$$\int \csc hx dx = 2\tanh^{-1} e^x$$

$$\int \sinh^2 x dx = \frac{1}{2}\left(-x + \frac{\sinh 2x}{2}\right)$$

$$\int \cosh^2 x dx = \frac{1}{2}\left(x + \frac{\sinh 2x}{2}\right)$$

$$\int \tanh^2 x dx = x - \tanh x$$

$$\int \coth^2 x dx = x - \coth x$$

$$\int \sec h^2 x dx = \tanh x$$

$$\int \csc h^2 x dx = -\coth x$$

$$\int \sec hx \tanh x dx = -\sec hx$$

$$\int \csc hx \coth x dx = -\csc hx$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$$

Properties of Definite Integrals

$$\int_a^b f(x)dx = \int_a^b f(t)dt$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

$$\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx --> f - even$$

$$\int_{-a}^a f(x)dx = 0 --> f - odd$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)dx$$

$$\int_0^{na} f(x)dx = n\int_0^a f(x)dx$$

if *f(x)* has a period ‘a’

Some special integrals:
Leibnitz rules:

$$\int_{a(x)}^{b(x)} f(t)dt = \int_C^{b(x)} f(t)dt - \int_C^{a(x)} f(t)dt$$

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(t)dt\right] = b'(x).f\left(b(x)\right) - a'(x).f\left(a(x)\right)$$

$$\frac{d}{da}\left[\int_{u(a)}^{v(a)} f(x,a)dx\right] = \int_{u(a)}^{v(a)} \frac{\delta}{\delta a} f(x,a)dx + v'(a).f\left(v(a),a\right) - u'(a).f\left(u(a),a\right)$$

Reduction formula for sin and cos functions:

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{(m-1)!!(n-1)!!}{(m+n)!!}$$

(if either m or n is odd)

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx = \frac{(m-1)!!(n-1)!!}{(m+n)!!}.\frac{\pi}{2}$$

(if both m and n are even)

$$m \geq 0, n \geq 0$$

NOTE: !!- can be called as **double/semi factorial**(though there is no such name in formal math literature).It symbolizes that we should **skip each number by 2 steps** and has been written just to simplify the formula.

Example:
Evaluate

$$\int_0^{\frac{\pi}{2}} \sin^7 x \cos^2 x dx$$

$$\int_0^{\frac{\pi}{2}} \sin^7 x \cos^2 x dx = \frac{(7-1)!!(2-1)!!}{(7+2)!!} = \frac{(6)!!(1)!!}{(9)!!}$$

$$= \frac{(6.4.2).(1)}{(9.7.5.3.1)} = \frac{48}{945}$$ [skipping each !! b 2 steps stop at 0 or 1]

{For $\int_0^{\frac{\pi}{2}} \sin^m x dx$ **or** $\int_0^{\frac{\pi}{2}} \cos^n x dx$
Put n=0or m=0. Do not take negative !! into account}

METHODS OF INTEGRATION

Method 1: Trignometric Simplification

$$2\sin A\cos B = \sin(A+B) + \sin(A-B)$$
$$2\cos A\sin B = \sin(A+B) - \sin(A-B)$$
$$2\cos A\cos B = \cos(A+B) + \cos(A-B)$$
$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}$$
$$\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \cos^2 A - \sin^2 A$$
$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$
$$\sin 2A = \frac{2\tan A}{1+\tan^2 A}$$
$$\cos 2A = \frac{1-\tan^2 A}{1+\tan^2 A}$$
$$\cos 3A = 4\cos^3 A - 3\cos A$$
$$\sin 3A = 3\sin A - 4\sin^3 A$$
$$\tan 3A = \frac{3\tan A - \tan^3 A}{1-3\tan^2 A}$$

Method 2: Substitution

$$I = \int F(f(x))\, f'(x)dx$$

Put $f(x) = t \rightarrow f'(x)dx = dt$

$$\therefore\ I = \int F(t)\,dt$$

Method 3: Completion of squares

Integrals of the type:

$$\frac{1}{ax^2+bx+c}, \frac{1}{\sqrt{ax^2+bx+c}} \text{ and } \sqrt{ax^2+bx+c}$$

complete the squares and reduce the integrand to standard formulas.

Method 4: Extended form of completion of squares

Integrals of the type:

$$\frac{Ax+B}{ax^2+bx+c}, \frac{Ax+B}{\sqrt{ax^2+bx+c}} \text{ and } (Ax+B)\sqrt{ax^2+bx+c}$$

Put $Ax+B=l[diff\,(ax^2+bx+c)]+m$ where l,m are constants.

$\rightarrow Ax+B=l[2ax+b]+m$

By equating like terms and constants l and m can be found.

Example

$$\int (x+1)\sqrt{x^2-4x+1}dx$$

$$x+1=l(2x-4)+m \rightarrow l=\frac{1}{2}, m=3 \rightarrow \int (\frac{1}{2}(2x-4)+3)\sqrt{x^2-4x+1}dx$$

$$\rightarrow \frac{1}{2}\int (2x-4)\sqrt{x^2-4x+1}dx+3\int \sqrt{x^2-4x+1}dx$$

This can be solved using standard formulas.

Method 5: Partial fractions

Numerator should be 1 degree lesser than Denominator

Example

$$1.)\frac{1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

$$2.)\frac{2x-1}{(x+1)(x^2+2x+5)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+5}$$

$$3.)\frac{x-1}{(x-2)^2(x+1)}=\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{C}{x+1}$$

$$4.)\frac{x^2}{(4+x^2)(6+x^2)}=\frac{A}{4+x^2}+\frac{B}{6+x^2}$$

$$5.)\int\frac{x+2}{(x-1)^7}dx=?$$

let us make an **algebrical simplification**

let $x-1=u \rightarrow x=1+u \rightarrow x+2=3+u$

$$\rightarrow\frac{u+3}{u^7} \rightarrow \frac{1}{u^6}+\frac{3}{u^7}$$

$$\therefore\int\frac{x+2}{(x-1)^7}dx=\int\frac{1}{(x-1)^6}+\frac{3}{(x-1)^7}dx$$

etc

$$6.)\frac{ax^2+b}{cx+d}=quotient+\frac{remainder}{devisor}$$

Method 6:

Integrals of the type

$$I_1=\frac{1+x^2}{1+x^4}\quad I_2=\frac{1-x^2}{1+x^4}\quad I_3=\frac{x^2}{1+x^4}\quad I_4=\frac{1}{1+x^4}$$

To solve the above integrals we use the following:

$$d(x+\frac{1}{x})=1-\frac{1}{x^2}dx$$

$$d(x-\frac{1}{x})=1+\frac{1}{x^2}dx$$

$$(x+\frac{1}{x})^2=x^2+\frac{1}{x^2}+2$$

$$(x-\frac{1}{x})^2=x^2+\frac{1}{x^2}-2$$

$$I_1=\int\frac{1+x^2}{1+x^4}dx=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\rightarrow x-\frac{1}{x}=t$$

$$\rightarrow I_1=\int\frac{dt}{t^2+2}$$

This can be solved using standard formula.

Similarly we can find for I_2

Now

$$I_3=\frac{I_1-I_2}{2}$$

$$I_4=\frac{I_1+I_2}{2}$$

Method 7:

Integrals of the type $\int\frac{dx}{(Ax+B)\sqrt{ax^2+bx+c}}$

Put

$$Ax+B=\frac{1}{t}$$

Example:

$$\int \frac{dx}{(x+1)\sqrt{x^2+3x+1}}$$

$$x+1=\frac{1}{t} \rightarrow dx=-\frac{1}{t^2}dt$$

$$\rightarrow x=\frac{1}{t}-1$$

$$\int \frac{dx}{(x+1)\sqrt{x^2+3x+1}}=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}-1\right)^2+3(\frac{1}{t}-1)+1}}$$

$$=-\int \frac{dt}{\sqrt{1+t-t^2}}$$

This can be solved using Completion of Squares (Method 3)

Method 8:

For integrals of the type $\int \frac{dx}{Ax^2+B\sqrt{ax^2+b}}$

Put $x=\frac{1}{t}$

Example

$$\int \frac{dx}{(2x^2-1)\sqrt{x^2+1}} \rightarrow x=\frac{1}{t} \rightarrow dx=-\frac{1}{t^2}dt$$

$$\int \frac{\frac{-dt}{t^2}}{(\frac{2}{t^2}-1)\sqrt{\frac{1}{t^2}+1}}=-\int \frac{tdt}{(2-t^2)\sqrt{1+t^2}}$$

Now $1+t^2=u^2 \dashrightarrow tdt=udu$

$$\therefore -\int \frac{udu}{(2-u^2+1)u} = \int \frac{du}{u^2-3}$$

This can be solved using standard formula

Method 9:

Integrals of the type:

$$\int \frac{dx}{a+b\cos x}, \int \frac{dx}{a+b\sin x}, \int \frac{dx}{a+b\cos x+c\sin x}$$

Put $t=\tan\frac{x}{2}$

$\rightarrow dt=\frac{1}{2}.\sec^2\frac{x}{2}dx$

$\therefore dx=\frac{2dt}{\sec^2\frac{x}{2}}=\frac{2dt}{1+t^2}$

Now

$\sin x=\frac{2t}{1+t^2}$

$\cos x=\frac{1-t^2}{1+t^2}$

The integrals will now reduce to standard formulas.

Method 10:

Integrals if the type:

$$\int \frac{dx}{a\cos^2 x + b\sin^2 x + c}$$

Divide Nr and Dr by $\cos^2 x$ or multiply by $\sec^2 x$
Then put $t = \tan x$
This will reduce to standard formulas.

Method 11:

Integrals of the type:

$$\int \frac{a\cos x + b\sin x}{A\cos x + B\sin x} dx$$

Write $Nr = l(dr) + m(differential(dr))$
This will reduce to standard formulas.

Method 12:

Integrals of the type:

$$\int \frac{a\cos x + b\sin x + c}{A\cos x + B\sin x + C} dx$$

Write $Nr = l(dr) + m(differential(dr)) + n$
This will reduce to standard formulas.

Method 13:

Integrals in which
$(\sin x \pm \cos x)$ appears in Numerator and
$f(\sin x.\cos x)$ appears in Denominator

Put either $\sin x + \cos x = t$ or $\sin x - \cos x = t$

Example

$$\int \left(\sqrt{\tan x} + \sqrt{\cot x}\right) dx = \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}}$$

Now let $\sin x - \cos x = t$

$\rightarrow \quad \cos x + \sin x = dt$

$\therefore \quad t^2 = 1 - 2\sin x \cos x$

$\rightarrow \quad \sin x \cos x = \dfrac{1-t^2}{2}$

$\rightarrow \quad \displaystyle\int \frac{dt}{\sqrt{\frac{1-t^2}{2}}}$

This can be solved using standard formulas.

Method 14: Integration by parts

Integrals of the type:

$$\int u dv = uv - \int v du$$

where dv-easier to integrate,u-difficult to integrate(hence differentiate)

Priority for u
ILATE
I-inverse trigonometric function
L-logrythmic function
A-algebric function
T-trignometric function
E-exponential function

Simplification of Integration by parts → BERNOULLI

$$\int udv = (u)(v_1) - (u')(v_2) + (u'')(v_3) - (u''')(v_4) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

where $u, u', u'' \ldots$ **represent the differentials and**
$v_1, v_2, v_3 \ldots$ **represent the integrals**

Best method to solve problems with x^n

Example

$$\int x^3 \sin x dx = (x^3)(-\cos x) - (3x^2)(-\sin x) + (6x)(\cos x) - (6)(\sin x)$$

Method 15: Standard Substitution:

$$f\left(\sqrt{a^2 - x^2}\right) \rightarrow x = a\sin t \,/\, a\cos t$$

$$f\left(\sqrt{a^2 + x^2}\right) \rightarrow x = a\tan t$$

$$f\left(\sqrt{\frac{a+x}{a-x}}\right) \rightarrow x = a\cos 2t$$

$$f\left(\sqrt{x^2 - a^2}\right) \rightarrow x = a\sec t$$

$$f\left(\sqrt{F(x)}\right) \rightarrow F(x) = t^2$$

Method 16:

Integrals of the type:

$$\int \frac{dx}{\cos(x-a)\cos(x-b)}, \int \frac{dx}{\sin(x-a)\sin(x-b)}$$

$$\int \frac{dx}{\sin(x-a)\cos(x-b)}, \int \frac{dx}{\cos(x-a)\sin(x-b)}$$

$$\frac{1}{\cos(x-a)\cos(x-b)}=\frac{1}{\sin(b-a)}\frac{\sin((x-a)-(x-b))}{\cos(x-a)\cos(x-b)}$$
$$=\frac{1}{\sin(b-a)}\frac{\sin(x-a)\cos(x-b)-\cos(x-a)\sin(x-b)}{\cos(x-a)\cos(x-b)}$$
$$=\frac{1}{\sin(b-a)}\left[\tan(x-a)-\tan(x-b)\right]$$

This can be solved using standard formulas.

Similarly other types can also be factorized.

Method 17:

Integrals of the type:

$$\int_a^b\sqrt{\frac{x-a}{b-x}}dx\ ,\ \int_a^b\sqrt{(x-a)(x-b)}dx\ ,\ \int\frac{1}{\sqrt{(x-a)(b-x)}}dx$$

Consider

$$\int_a^b\sqrt{\frac{x-a}{b-x}}dx$$

put

$$x=a\cos^2 t+b\sin^2 t$$
$$\therefore x-a=a\cos^2 t+b\sin^2 t-a=b\sin^2 t-a\sin^2 t=(b-a)\sin^2 t$$
$$\therefore b-x=b-a\cos^2 t+b\sin^2 t=b\cos^2 t-a\cos^2 t=(b-a)\cos^2 t$$
$$\to\ dx=\left(-2a\cos t\sin t+2b\sin t\cos t\right)dt=2(b-a)\sin t\cos t dt$$
$$\to\ x=a\ \to\ t=0, x=b\ \to\ t=\frac{\pi}{2}$$
$$\to\ 2(b-a)\int_0^{\frac{\pi}{2}}\sin^2 t dt=\frac{\pi}{2}(b-a)$$

BETA AND GAMMA FUNCTIONS

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$
$$\longleftrightarrow m>0, n>0$$
$$\beta(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$
$$\beta(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}x\cos^{2n-1}x dx$$
$$\beta(m,n)=\beta(n,m)$$
$$\beta(m,n)=\beta(m+1,n)+\beta(m,n+1)$$

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$$
$$\longleftrightarrow n>0$$
$$\Gamma(n+1)=n\Gamma(n)$$
$$\Gamma(n+1)=n!$$
$$\Gamma(n)=2\int_0^\infty e^{-y^2}y^{2n-1}dy$$
$$\Gamma(n)=\int_0^1\left[\log_e\frac{1}{x}\right]^{n-1}dx$$
$$2^{2n-1}\Gamma(n)\Gamma(n+\frac{1}{2})=\Gamma(2n)\sqrt{\pi}$$
$$\Gamma(\frac{1}{2})=\sqrt{\pi}$$
$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$
$$\int_0^\infty \frac{x^{m-1}}{(1+x)}dx=\beta(m,1-m)=\Gamma(m)\Gamma(1-m)$$
$$2\int_0^{\frac{\pi}{2}}\sin^p x\cos^q x dx=\beta(\frac{p+1}{2},\frac{q+1}{2})$$

if p is a negative fraction

$$\Gamma(p) = \frac{1}{p}\Gamma(p+1)$$